5. Varieties

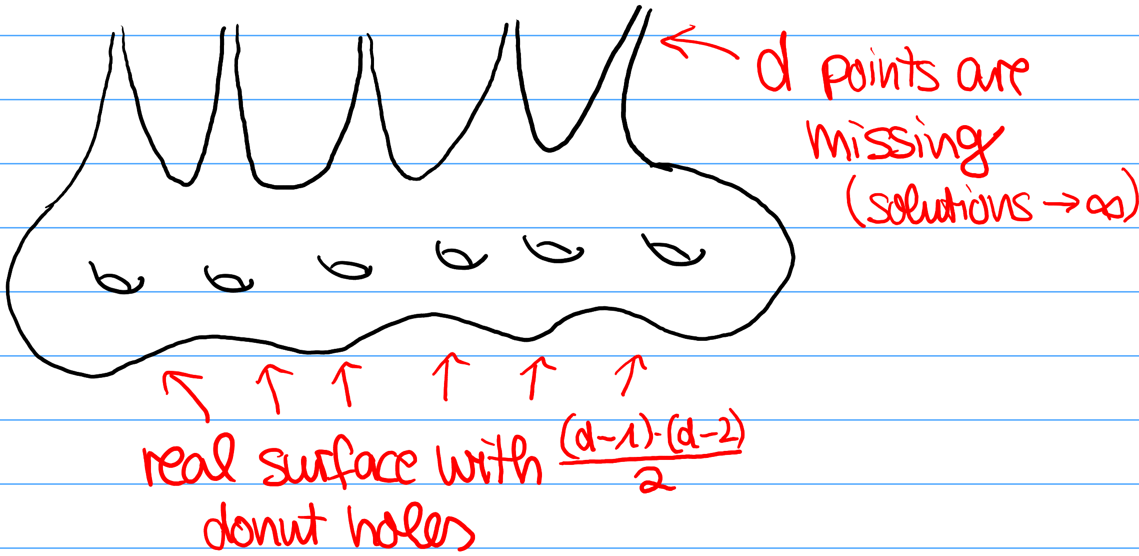
Big picture

- Have introduced a nice class of ringed spaces: affine var. X
- These have some limitations
  - → open subsets of aff. var are not necess. aff. var (e.g. $\mathbb{A}^2 \setminus \{0\} \subseteq \mathbb{A}^2$)
  - → for $K = \mathbb{C}$, the complex topology on X is only compact if X = finite

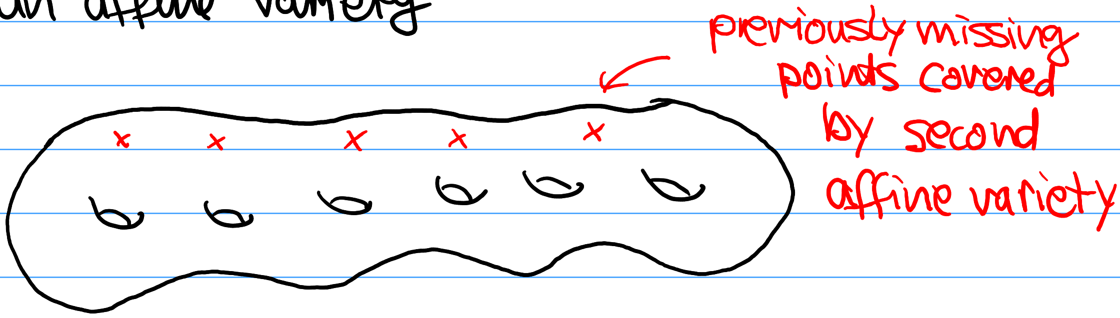
    Exa inside $\mathbb{A}^2_{\mathbb{C}} = \mathbb{C}^2$, a general hypersurface (coefficients chosen generically)

    $X = V(f)$, $f \in K[x,y]_d$ of degree d

    looks as follows:



  - → solution: look at ringed spaces which locally look like an affine variety



(analogy: manifolds = top. spaces which locally look like open subsets of $\mathbb{R}^n$)

Def (Prevarieties)

A **prevariety** is a ringed space X that has a finite open cover by affine varieties. Morphisms of prevarieties are simply morphisms of ringed spaces.

For $U \subseteq X$ open: elements of $\mathcal{O}_X(U)$ $=:$ **regular functions**

Rmk • open cover above is not part of the data of a prevariety (enough that it exists)

• An open subset of a prevariety that is an affine variety is called an **affine open set**.

Exa • Any affine variety is a prevariety.

• $U = \mathbb{A}^2 \setminus \{0\} \subseteq \mathbb{A}^2$ is a prevariety ($U = D(x_1) \cup D(x_2)$) but not affine.

• More generally: any open subset U in an affine variety X is a prevariety

$\rightsquigarrow$ finite cover by distinguished open sets $D(f)$, which are themselves affine varieties.

basis of topology

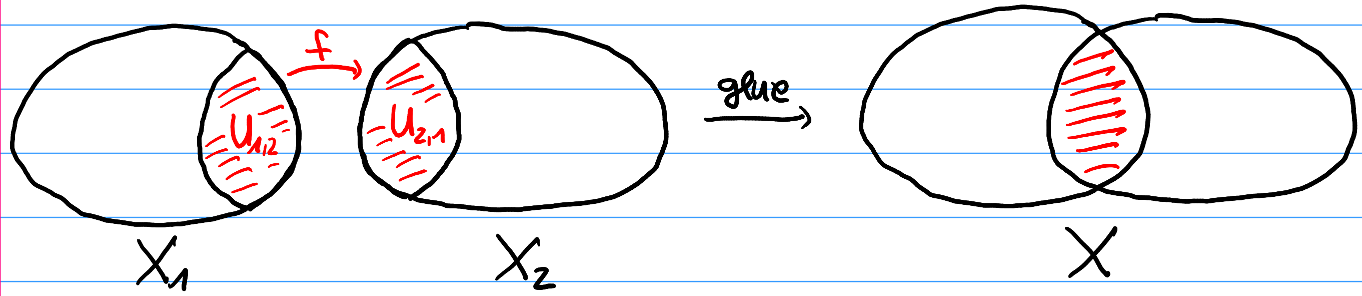
Before we complained that aff. var. are not closed under taking open subsets. This is solved with the new definition.

Exercise Show that any open subset of a prevariety is a prevariety.

Gluing two prevarieties

Basic construction of new prevarieties: patch them together from affine varieties (or more generally: prevarieties)

Construction (Gluing two prevarieties)
Let X_1, X_2 be two prevarieties (e.g. aff. var.) and let $U_{1,2} \subseteq X_1$ and $U_{2,1} \subseteq X_2$ be open subsets.
Assume we have an isomorphism $f: U_{1,2} \xrightarrow{\sim} U_{2,1}$.
Then we can define a prevariety X obtained by <u>gluing</u> X_1 and X_2 along f as shown below:



To obtain ringed space X:

- As a <u>set</u>: $X = X_1 \sqcup X_2 / \sim$ where $\sim$ is the equivalence relation given by $a \sim f(a)$, $f(a) \sim a$ (for $a \in U_{1,2}$)
  $x \sim x$ (for $x \in X_1 \sqcup X_2$)

  This gives two injective maps $i_1: X_1 \to X$, $i_2: X_2 \to X$ mapping x to its equivalence class.
- As a <u>topological space</u>: take quotient topology from map

$$\underbrace{X_1 \sqcup X_2}_{\text{has a topology}} \xrightarrow{i_1 \sqcup i_2} \underbrace{X}_{\text{put finest topology making } i_1 \sqcup i_2 \text{ continuous}}$$

Explicitly: $U \subseteq X$ is open if and only if $i_1^{-1}(U) \subseteq X_1$ open and $i_2^{-1}(U) \subseteq X_2$ open.

$\rightsquigarrow$ i_1, i_2 are homeomorphisms onto their images
$\Rightarrow$ Can see X_1, X_2 as open subsets of X.

• As ringed spaces: define structure sheaf $\mathcal{O}_X$ by

$$\mathcal{O}_X(U) = \{ \varphi : U \to K : i_1^* \varphi \in \mathcal{O}_{X_1}(i_1^{-1}(U)) \text{ and } i_2^* \varphi \in \mathcal{O}_{X_2}(i_2^{-1}(U)) \}$$

for $U \subseteq X$ open. ⇝ function on U is regular iff its restriction to $X_1 \cap U$ and $X_2 \cap U$ is.

⇝ this satisfies the sheaf axioms.

Note For the open sets $X_1, X_2 \subseteq X$ it follows: $\mathcal{O}_X|_{X_i} = \mathcal{O}_{X_i}$

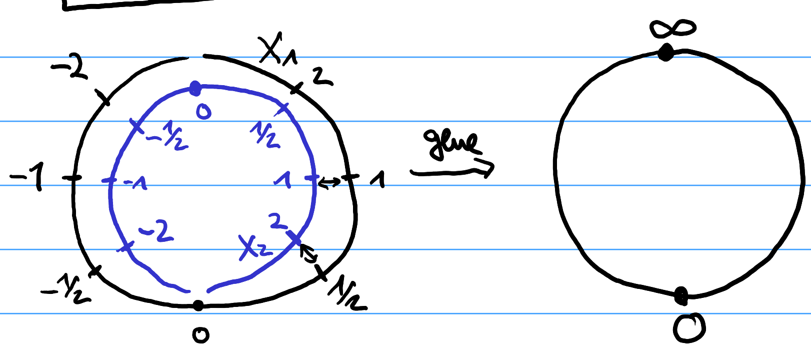
X_1, X_2 covered by aff. var $\Rightarrow$ same for X $\Rightarrow$ X is prevariety.

Two ways of gluing two affine lines

Exa Let $X_1 = X_2 = \mathbb{A}^1$ and $U_{1,2} = U_{2,1} = \mathbb{A}^1 \setminus \{0\}$.

Consider two different choices for $f : U_{1,2} \xrightarrow{\sim} U_{2,1}$.

(a) $\boxed{f(x) = 1/x}$ isomorphism w/ $f^{-1} = f$.



As a set:

$X = X_1 \cup \underbrace{(X_2 \setminus U_{2,1})}_{\text{single point} =: \infty}$

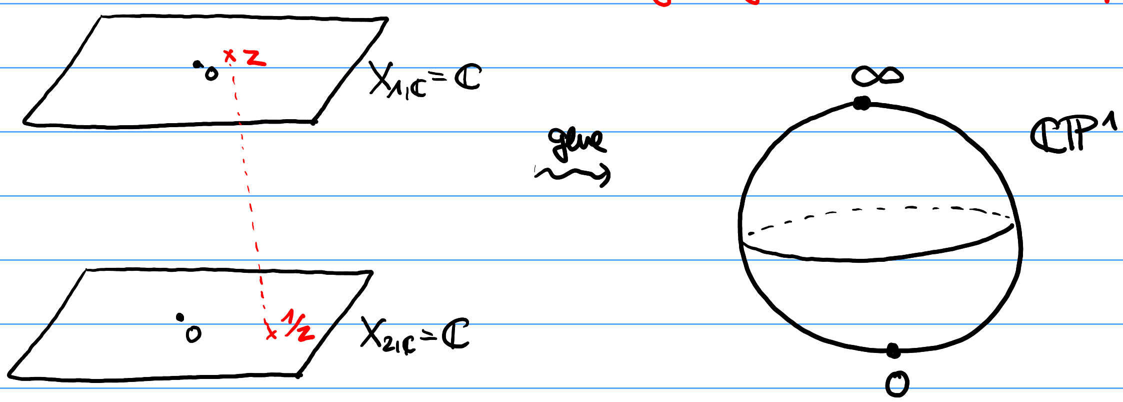
$\hat{=} (0 \in X_2)$

idea: $(0 \in X_2)$ $\hat{=} (1/0 = \infty) \text{ in } X_1$

⇝ resulting space $X =: \mathbb{P}^1$, the projective line (more: next chapters)

$K = \mathbb{C}$ ⇝ $\mathbb{P}^1_{\mathbb{C}} =$ Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$

could do same gluing operations for complex topology.



Can construct a map:

$$\begin{tikzcd}
\mathbb{P}^1 = X_1 \arrow[r, bend left, "z \mapsto 1/z"] \arrow[rrr, bend right, "x \mapsto x"'] & \cup\, X_2 \arrow[r, "f"] \arrow[r, bend left=50, "x \mapsto x"] & X_1 \arrow[r, bend left, "z \mapsto 1/z"] & \cup\, X_2 = \mathbb{P}^1
\end{tikzcd}$$

Check: morphisms agree on overlaps.
$\Downarrow$ Lem
morph. of ringed spaces glue.

This extends the map $X_1 \setminus \{0\} \longrightarrow X_1$, $x \mapsto 1/x$.

$\rightsquigarrow \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$, $f(42) = \frac{1}{42}$, $f(0) = \infty$, $f(\infty) = 0$

(b) $\boxed{f(x) = x}$ still have $X = X_1 \cup \{pt\}$ but now pt is like "a second origin"

$\rightsquigarrow$ X is the "affine line with two zero points"

<u>X is a weird space</u>

$\rightarrow$ gluing operation over cplx numbers $\rightsquigarrow X_{\mathbb{C}}$ is non-Hausdorff
(sequence $a_n = 1/n$ converges to <u>both</u> origins)

$\rightarrow$ similar gluing as above:

$$\begin{tikzcd}
X = X_1 \arrow[r, bend left, "z \mapsto z"] \arrow[rrr, bend right, "x \mapsto x"'] & \cup\, X_2 \arrow[r, "g"] \arrow[r, bend left=50, "x \mapsto x"] & X_1 \arrow[r, bend left, "z \mapsto z"] & \cup\, X_2 = X
\end{tikzcd}$$

$\rightsquigarrow$ obtain $g: X \to X$ $\begin{cases} \text{exchanging the two origins} \\ \text{identity away from two origins} \end{cases}$

$\Rightarrow$ the set

$$\{x \in X : g(x) = x\} = \mathbb{A}^1 \setminus \{0\} \subseteq X$$

is not closed (even though it is given by equality of morphisms g & id_X)

Later: define varieties as prevarieties + extra conditions to prevent such behaviour.

Gluing finite collections of prevarieties

Patching together three or more prevarieties works similar to the case of two, except that we need one more compatibility condition.

Construction (General gluing construction)

For a finite index set I let X_i be a prevariety for all $i \in I$.
Moreover suppose that for all $i,j \in I$ w/ $i \neq j$ we have open subsets

$$U_{ij} \subseteq X_i$$

and isomorphisms

$$f_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$$

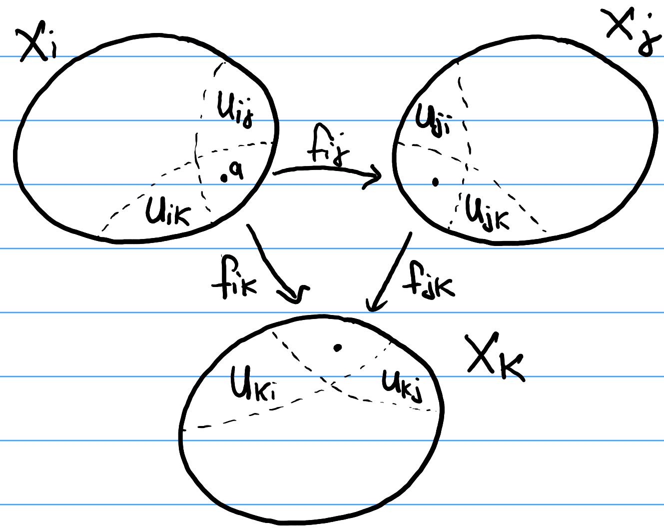
such that for all distinct $i,j,k \in I$:

(a) $f_{ji} = f_{ij}^{-1}$

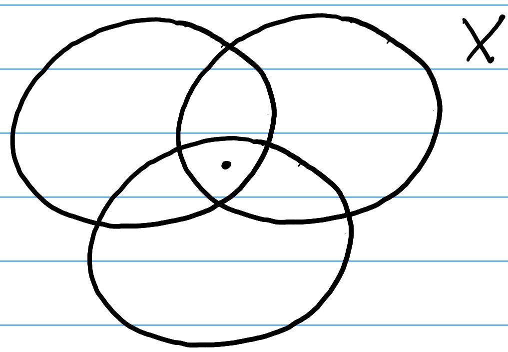
(b) $f_{ij}^{-1}(U_{jk}) \subset U_{ik}$ and

$$f_{jk} \circ f_{ij} = f_{ik} \text{ on } f_{ij}^{-1}(U_{jk})$$

Slogan: Transition map $X_i \rightsquigarrow X_j \rightsquigarrow X_k$ equals the transition map $X_i \rightsquigarrow X_k$.



glue



Then we set $X = \bigsqcup_{i \in I} X_i / \sim$ — $a \sim f_{ij}(a)$ for $a \in U_{ij}$, $x \sim x$ for $x \in X_i$

Conditions (a) & (b) $\Rightarrow$ $\sim$ is symmetric & transitive.
$\rightsquigarrow$ define topology & structure sheaf as in case of two prevarieties
$\Rightarrow$ X becomes a prevariety obtained by gluing the X_i along the maps f_{ij}
Have $X_i \subseteq X$ as open subsets, $X = \bigcup_{i \in I} X_i$.

<u>Exa</u> (Projective plane)

$U_0 = U_1 = U_2 = \mathbb{A}^2$ $\qquad U_{ij} = \{x \in U_i : x_j \neq 0\}$ use the coordinates specified in (*)

(*) $\underset{(x_1,x_2)}{U_0} \quad \underset{(x_0,x_2)}{U_1} \quad \underset{(x_0,x_1)}{U_2}$

$f_{01}(x_1,x_2) = \left(\frac{1}{x_1}, \frac{x_2}{x_1}\right)$, $f_{02}(x_1,x_2) = \left(\frac{1}{x_2}, \frac{x_1}{x_2}\right)$, $f_{12}(x_0,x_1) = \left(\frac{x_0}{x_1}, \frac{1}{x_1}\right)$

<u>Exercise</u> Check that the above gluing data is well-defined.

Later: Interpretation of glued prevariety $X = \mathbb{P}^2 = U_0 \cup U_1 \cup U_2$.

<u>Note</u> by definition any prevariety X has finite cover by affine varieties $X_1, \dots, X_m$

$\rightsquigarrow$ setting $U_{ij} = X_i \cap X_j \subseteq X_i$ and $U_{ij} \xrightarrow{id = f_{ij}} U_{ji}$ we recover X by gluing together the X_i

$\Rightarrow$ The above construction can recover <u>any</u> prevariety X from affine varieties X_i.

Basic properties and constructions of prevarieties

Topological properties & open and closed subprevarieties

<u>Before</u>: How to glue affine varieties together to form prevarieties.

<u>Next</u>: define & study basic properties (both generaliz. from affine varieties & new ones)

Topological properties

X prevariety $\rightsquigarrow$ X topological space

↳ open & closed subsets

↳ connectedness

↳ irreducibility

↳ dimension

$\left.\begin{array}{l} \\ \end{array}\right\}$ defined for arbitr. top. spaces

X has finite cover by aff. varieties $\xRightarrow{\text{[Gathmann Exc. 2.19 (b)]}}$ X is Noetherian

Noetherian

$\Rightarrow X$ has an irreducible decomposition.

Open & closed subprevarieties

<u>Construction</u> Let X be a prevariety

(a) Let $U \subseteq X$ be an open subset. Then U is again a prevariety (with $\mathcal{O}_U = \mathcal{O}_X|_U$ as usual).

Indeed:

$$X = \bigcup_{i \in I} \underset{\text{aff. var}}{X_i} \quad \Rightarrow \quad U = \bigcup_{i \in I} \underbrace{X_i \cap U}_{\text{open subset of aff. var. } X_i}$$

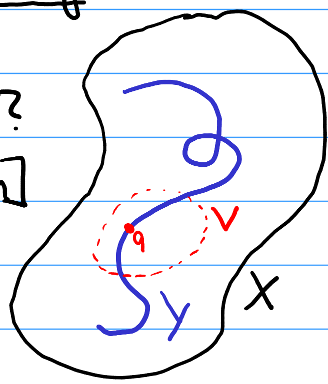
$\Rightarrow$ has finite cover by distinguished opens, which are affine varieties themselves

We call U an <u>open subprevariety</u> of X and the morphism $i_U : U \hookrightarrow X$ an <u>open embedding</u>.

(b) Let $Y \subseteq X$ be a closed subset

$\rightsquigarrow$ how to define $\mathcal{O}_Y(U)$ for $U \subseteq Y$ open?

[Note: cannot just take $\mathcal{O}_X(U)$ since $U \subseteq X$ not necess. open]



<u>Idea</u> Define $\mathcal{O}_Y(U)$ as functions $U \to K$ that are locally restrictions of regular functions on X.

$$\mathcal{O}_Y(U) = \left\{ \varphi : U \to K \;\middle|\; \begin{array}{l} \text{for all } a \in U \text{ there are an open nbhd. } V \\ \text{of } a \text{ in } X \text{ and } \psi \in \mathcal{O}_X(V) \text{ with } \varphi = \psi \text{ on } U \cap V \end{array} \right\}$$

$\rightsquigarrow$ $\mathcal{O}_Y$ gives sheaf of K-algebras (since cond. on φ is local)

$\rightsquigarrow$ $(Y, \mathcal{O}_Y)$ is a ringed space

<u>Exercise</u> Show that for any affine open $W \subseteq X$:

$$(W \cap Y, \mathcal{O}_Y|_W) \xrightarrow{\text{id}} (W \cap Y, \underbrace{\mathcal{O}_{W \cap Y}}_{W \text{ affine},\ W \cap Y \subseteq W \text{ closed} \Rightarrow W \cap Y \text{ is affine var.}})$$ is an isom. of ringed spaces.

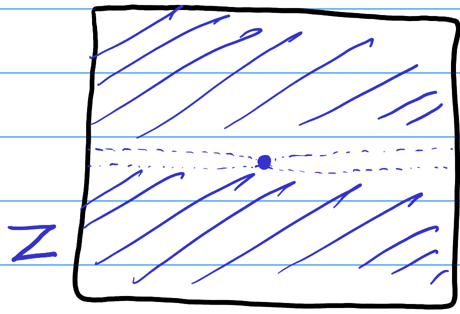
$\Rightarrow$ $(Y, \mathcal{O}_Y)$ has fin. cover by aff. varieties $\rightsquigarrow$ prevariety

We call Y a <u>closed subprevariety</u> of X.

Rmk For general subsets $Z \subseteq X$ of a prevariety there does not necessarily exist a good structure of a prevariety on Z.

Exa In $X = \mathbb{A}^2$ take

$$Z = \underbrace{\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})}_{\text{open in } X} \cup \underbrace{\{(0,0)\}}_{\text{closed in } X}$$



$\rightsquigarrow$ not naturally a prevariety
(e.g. does not look like an aff. var. around $(0,0)$)

The above example shows that we shouldn't mix open and closed subsets of X.... For more info see

Digression: Locally closed & constructible sets

Exercise (Properties of closed subprevarieties, [Gath., Rmk 5.12])
Let $Y \subseteq X$ be a closed subprevariety
(a) The inclusion $i_Y : Y \hookrightarrow X$ is a morphism → call such morphisms closed embeddings
(b) For $f : Z \to X$ a morphism from a prevariety Z with $f(Z) \subseteq Y$, also $Z \to Y$, $z \mapsto f(z)$ is a morphism.

Rmk (Images & inverse images of subprevarieties)
Let $f : X \to Y$ be a morphism of prevarieties.
(a) The image of an open/closed subprevar. in X is not necessarily an open/closed subprevar. in Y

Exa $f : \mathbb{A}^2 \to \mathbb{A}^2$, $(x,y) \mapsto (xy, y)$ satisfies

neither open nor closed

$$f(\mathbb{A}^2) = Z = \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\}) \cup \{(0,0)\} \subseteq \mathbb{A}^2$$

both open and closed

Proof For $U = \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$: $f|_U : U \xrightarrow{\sim} U$, inverse: $(u,v) \mapsto (u/v, v)$
$f(\mathbb{A}^1 \times \{0\}) = \{(0,0)\}$.

(b) f continuous $\rightsquigarrow$ preimage of $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ subprevar. is $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$.

Products of prevarieties

We know: U, V affine varieties $\Rightarrow$ $U \times V$ affine variety

Q What about products of prevarieties X and Y?

Idea 1 (Existence)

- Choose affine open covers $\{U_i : i \in I\}, \{V_j : j \in J\}$ of X, Y
- Glue the affine varieties $U_i \times V_j$ $((i,j) \in I \times J)$ to obtain $X \times Y$

Problem How to show result is independent of choice of covers?

Idea 2 (Uniqueness)

Characterize the product $X \times Y$ by a universal property (& check for construction above).

Def (Products of prevarieties)

Let X, Y be prevarieties. A product of X and Y is a prevariety P together with morphisms $\pi_X : P \to X$ and $\pi_Y : P \to Y$ satisfying the following universal property:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "f"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_Y"'] & & \\
& P \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \\
& Y &
\end{tikzcd}$$

note: this makes sense in any category!

(*) For any two morphisms $f_X : Z \to X$ and $f_Y : Z \to Y$ from another prevariety Z there is a unique morphism $f : Z \to P$ such that $f_X = \pi_X \circ f$ and $f_Y = \pi_Y \circ f$.

Slogan Giving a morphism f to the product of X, Y is the same data as giving a pair of morphisms f_X, f_Y to X, Y.

In [Gath., Pro 4.10] we checked that for X, Y affine, property (*) holds for $P = X \times Y$ when Z is affine.

Pro (Existence and uniqueness of products)
Any two prevarieties X and Y have a product (P, π_X, π_Y).
Moreover, it is unique up to unique isomorphism:
If (P', π_X', π_Y') is another product, there is a unique isomorphism $f: P' \to P$ such that
$\pi_X \circ f = \pi_X'$ and $\pi_Y \circ f = \pi_Y'$. $(**)$

$$\begin{tikzcd} P \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, dashed, "g"] & X \\ Y & P' \arrow[l, "\pi_Y'"] \arrow[u, "\pi_X'"'] \arrow[ul, dashed, "f"'] \end{tikzcd}$$

We denote this product by $P = X \times Y$.

Proof $\boxed{\text{Uniqueness}}$
- universal prop. of P for $Z = P'$ $\begin{smallmatrix} \xrightarrow{\pi_X'} X \\ \xrightarrow[\pi_Y']{} Y \end{smallmatrix}$ $\rightsquigarrow$ morphism $f: P' \to P$ — unique one with $(**)$
- symmetric argument for P' $\rightsquigarrow$ morphism $g: P \to P'$

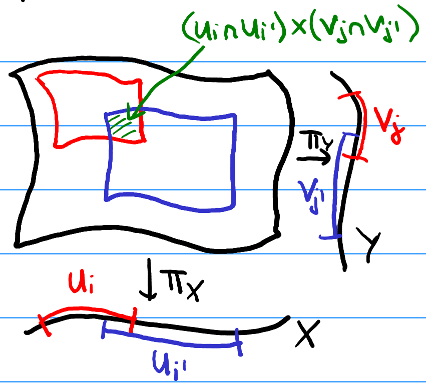
Claim $g \circ f = id_{P'}$ and $f \circ g = id_P$ $\rightsquigarrow$ this implies f isom & finishes uniqueness!

Pf. of claim: $\pi_X' \circ (g \circ f) \overset{\text{Def of } g}{=\!=} \pi_X \circ f \overset{\text{Def of } f}{=\!=} \pi_X' = \pi_X' \circ id_{P'}$
similar: $\pi_Y' \circ (g \circ f) = \pi_Y' \circ (id_{P'})$

$\rightsquigarrow$ $P' \underset{id_{P'}}{\overset{g \circ f}{\rightrightarrows}} P'$ two morphisms with same composit. with π_X', π_Y'

$\xrightarrow[\text{of } P']{\text{univ. prop.}}$ $g \circ f = id_{P'}$ (similar: $f \circ g = id_P$).

$\boxed{\text{Existence}}$ We construct P explicitely
$X = U_1 \cup \dots \cup U_n$ and $Y = V_1 \cup \dots \cup V_m$ affine open covers
$\rightsquigarrow$ glue products $U_i \times V_j$ along the identity map on their overlaps $(U_i \cap U_{i'}) \times (V_j \cap V_{j'})$



identity maps satisfy (a) & (b) from gluing construction $\rightsquigarrow$ glued prevar. P — $P = X \times Y$ as a set
Project. morphisms $U_i \times V_j \to U_i \subseteq X$
$U_i \times V_j \to V_j \subseteq Y$
glue to morphisms $\pi_X: P \to X$ and $\pi_Y: P \to Y$

$\boxed{\text{Check of universal property}}$ Have candidate (P, π_X, π_Y)

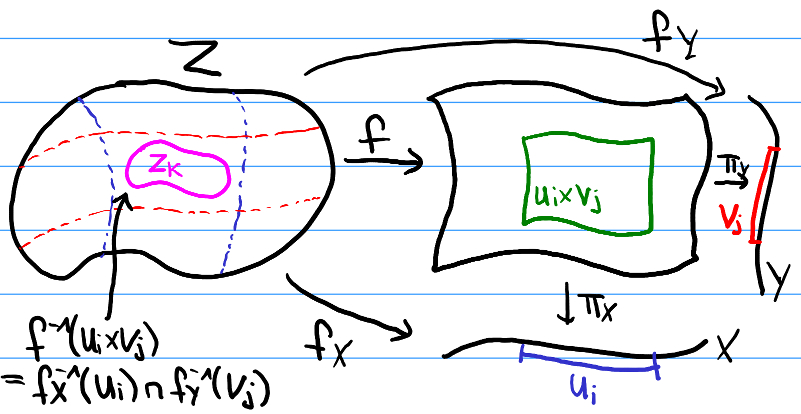
Let Z be a prevariety and $f_X : Z \to X$, $f_Y: Z \to Y$ morphisms

$\rightsquigarrow$ only chance to get $f: Z \to P$ with $(*)$ is

$$f(z) = (f_X(z), f_Y(z)) \in X \times Y \overset{\text{as set}}{=\!=} P$$

Need to check: f is a morphism.

[Lem. 4.6] can check this on an open cover of Z



$\to$ first take open sets $f^{-1}(U_i \times V_j) \subseteq Z$

$\to$ then cover those by fin. many affine opens Z_K

$\rightsquigarrow$ $f_X|_{Z_K} : Z_K \to U_i \subseteq X$ and $f_Y|_{Z_K} : Z_K \to V_j \subseteq Y$ are morphisms

[Pro 4.10] $\Rightarrow$ $f|_{Z_K} : Z_K \to U_i \times V_j$ is a morphism

$$\begin{array}{c} U_i \times V_j \\ \downarrow \\ P \end{array} \leftarrow \text{is morphism by gluing construction}$$

Since the Z_K cover all of P

$\overset{\text{[Lem 4.6]}}{\Longrightarrow}$ f is a morphism. $\square$

Rmk X, Y prevarieties, $X' \subseteq X$ and $Y' \subseteq Y$ closed subprevarieties

$\Rightarrow$ The set $X' \times Y'$ has two prevariety-structures

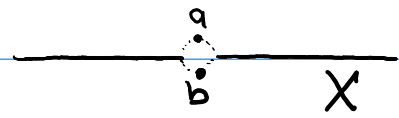
$\Rightarrow$ identity gives inverse isomorphisms: $\underbrace{X' \times Y'}_{\substack{\text{product of} \\ X' \text{ and } Y'}} \longleftrightarrow \underbrace{X' \times Y'}_{\substack{\text{subprevariety} \\ \text{of } X \times Y}} \subseteq X \times Y$

(use [Rmk 5.12] & univ. prop. $(*)$)

Separatedness and the definition of varieties

Big picture

- prevarieties include strange spaces like affine line X with two origins $a, b \in X$



- bad properties (e.g. morphisms $f, g : \mathbb{A}^1 \to X$ w/ $\{x : f(x) = g(x)\} \subseteq \mathbb{A}^1$ not closed!
- in complex topology: $X_{\mathbb{C}}$ is not Hausdorff
- in Zariski topology: want analogue of Hausdorff condition

  Hausdorff itself too strong!

- **Pro** (Topology) Top. space X is Hausdorff if and only if the diagonal $\Delta_X = \{(a,a) : a \in X\} \subseteq X \times X$ is closed.

  product topology

- Try the same def. for X variety (w/ Zar. top. on $X \times X$)

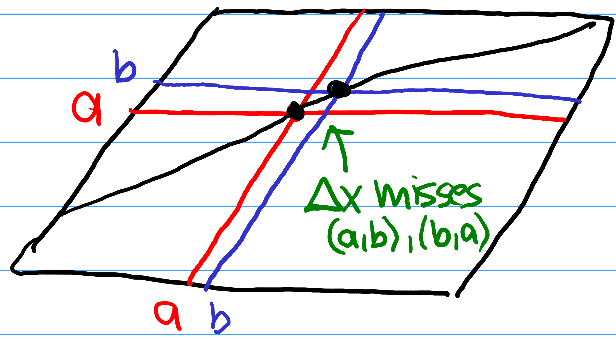
Exa For X the aff. line w/ two origins $a, b \in X$

Claim

Zariski closure

$$\overline{\Delta_X} = \Delta_X \cup \{(a,b), (b,a)\}$$

$\Rightarrow \Delta_X$ is not closed!



Pf of claim: Recall open cover $X = U_a \cup U_b$, $U_a = \mathbb{A}^1$, $U_b = \mathbb{A}^1$

$$\rightsquigarrow \Delta_X \cap \underbrace{U_a \times U_b}_{\subseteq X \times X} = \underbrace{\{(x,y) \in \mathbb{A}^2 : x = y, x \neq 0\}}$$

taking closure adds point $(x,y) = (0,0)$

$\rotatebox{90}{$\mapsto$}$

$X \times X \ni (a,b)$

Def (Varieties)

A prevariety X is called a <u>variety</u> (or <u>separated</u>) if the <u>diagonal</u>

c.f. separation axioms in topology (Hausdorff $= T_2$)

$$\Delta_X = \{(x,x) : x \in X\} \subseteq X \times X$$

is closed in $X \times X$.

Rmk

- calculation above: affine line w/ two origins is <u>not</u> separated / variety
- results below: many prevarieties we naturally encounter are varieties & varieties have nice properties

$\rightsquigarrow$ mostly consider varieties from now on.

Lem (a) Affine varieties are varieties.

(b) Open & closed subprevarieties of varieties are varieties.

just call them <u>subvarieties</u>

Proof

(a) If $X \subseteq \mathbb{A}^n$ then $\Delta_X = V_{X\times X}(x_1 - y_1, \dots, x_n - y_n) \subseteq X \times X \subseteq \mathbb{A}^n \times \mathbb{A}^n$

$(x_1 \dots x_n, y_1 \dots y_n)$ coordinates

$\rightsquigarrow \Delta_X$ is closed

(b) $Y \subseteq X$ open / closed $\rightsquigarrow$ try to show $\Delta_Y \subseteq Y \times Y$ closed using univ. property!

Claim: The map $i_{Y\times Y} : Y \times Y \to X \times X$, $(y_1, y_2) \mapsto (y_1, y_2)$ is a morphism.

Then: $\Delta_Y = i_{Y\times Y}^{-1}(\Delta_X)$ is closed since $i_{Y\times Y}$ is continuous, $\Delta_X \subseteq X \times X$ closed.

Pf. of claim:

By univ. prop. of $X \times X$: have to show $Y \times Y \xrightarrow[(y_1,y_2)\mapsto y_1]{i_1} X$ is morphism (similar: i_2)

$$\begin{tikzcd} Y \times Y \arrow[r, "i_1", "(y_1,y_2)\mapsto y_1"'] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\mathrm{I}"] & X \\ Y \arrow[ur, bend right, "i_Y"'] & \end{tikzcd}$$

open/closed embedding.

$\Rightarrow i_1 = i_Y \circ \pi_1$ is morphism as composit. of morphisms

$\Rightarrow i_{Y\times Y} = (i_1, i_2)$ is morphism. $\square$

Exercise (Products of morphisms)
Let $f_1: Y_1 \to X_1$ and $f_2: Y_2 \to X_2$ be morphisms.
(a) Show that $(f_1, f_2): Y_1 \times Y_2 \to X_1 \times X_2$ is a morphism.
(b) Prove that if f_1, f_2 are open (or closed) embeddings, also (f_1, f_2) is an open (or closed) embedding.
(Hint: Show that being an open/closed embedding can be checked on an open cover of the target.)

Def. A variety of pure dimension 1 or 2 is called a *curve* (resp. *surface*).
- If X is a pure-dim. variety and Y a pure-dim. closed subvariety of X with $\dim Y = \dim X - 1$, we call Y a *hypersurface* in X.

Pro (Properties of varieties)
Let $f, g: X \to Y$ be morphisms of prevarieties and assume that Y is a variety.
(a) The graph

$$\Gamma_f := \{(x, f(x)) : x \in X\} \subseteq X \times Y$$

is closed in $X \times Y$.
(b) The set $\{x \in X : f(x) = g(x)\}$ is closed in X.
Slogan: algebraic conditions cut out closed sets.

Pf (a) $(f, \mathrm{id}_Y): X \times Y \to Y \times Y$ morphism [Exercise] and $\Gamma_f = (f, \mathrm{id}_Y)^{-1} \Delta_Y$. Y variety $\Rightarrow \Delta_Y$ closed $\xRightarrow[\text{contin.}]{(f, \mathrm{id}_Y)} \Gamma_f$ closed.

(b) $(f, g): X \to Y \times Y$ morphism, $\{x : f(x) = g(x)\} = (f, g)^{-1} \Delta_Y$. $\square$

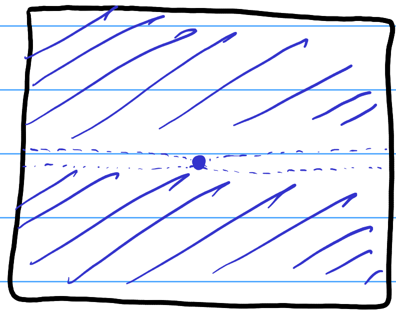
Exercise X, Y varieties $\Rightarrow X \times Y$ variety.

Digression: Locally closed & constructible sets

In lecture we saw: open & closed subprevarieties
⇝ there is a useful way to mix these:

Def (Locally closed & constructible sets)
Let X be a top. space. A subset $S \subseteq X$ is called
- locally closed if it is the intersection $S = U \cap C$ of an open set $U \subseteq X$ and a closed set $C \subseteq X$, in part.: U, C loc. closed.
- constructible if it is a finite union of locally closed sets.

Exa For $X = \mathbb{A}^2$ we have:
- $S = \{0\} \times (\mathbb{A}^1 \setminus \{0\}) \subseteq X$ is locally closed:
  $S = \underbrace{\{(x,y) : y \neq 0\}}_{U \text{ open}} \cap \underbrace{\{(x,y) : x = 0\}}_{C \text{ closed}}$
- The set $Z = \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\}) \cup \{(0,0)\}$ is constructible (but not loc. closed)



Z

Exercise X prevariety, $S = U \cap C \subseteq X$ locally closed
⇝ Can see S as ⇝ open subprevariety of C
⇝ closed subprevariety of U

Convince yourself that these two structures of prevariety of X are isomorphic.

In main lecture we saw that the image of an open/closed set under a morphism of prevarieties is not necess. open/closed.

Theorem (Chevalley's thm.)
For $f: X \to Y$ a morphism of prevarieties and $S \subseteq X$ constructible, its image $f(S) \subseteq Y$ is constructible.

Exa $f: \mathbb{A}^2 \to \mathbb{A}^2$, $(x,y) \mapsto (xy, y)$ satisfies $f(\mathbb{A}^2) = Z$ ⇜ from above